

INTERACTION OF GRAVITATIONAL WAVES WITH AN ELASTIC SOLID MEDIUM

B. Carter

Groupe d'Astrophysique Relativiste
Observatoire de Paris
92 Meudon, France

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1 INTRODUCTION

Although the presently most promising approach to the direct detection of gravitational radiation would seem to be provided by the use of appropriate electromagnetic field configurations (e.g. that of an optical interferometer) the original, and until now most highly developed method has been based on the use of an elastic solid configuration, traditionally a cylindrical bar of the kind originally introduced by Weber. A highly simplified but essentially adequate description of the interaction of gravitational radiation with such an apparatus was given by Weber himself in his seminal 1960 paper [1]. For a more detailed analysis, and for the treatment of more general configurations such as that of the earth as a whole, a system of wave equations governing the interaction of weak gravitational radiation with an elastic solid was derived soon after by Dyson [2], Papapetrou [3], and others, subject to the limitation that non-linearities due to self gravitation of the solid medium are neglected, as is entirely justifiable in a terrestrial context.

While the final outcome of the present article will be a rederivation of the weak field wave equations to which we have just referred, we shall nevertheless use a quite different approach to that of Dyson and Papapetrou. Instead of working throughout in an only approximately self consistent linearised scheme, as they did, we shall first set up the exactly self consistent fully non-linear theory of the interaction of a gravitational field with an elastic solid in accordance with Einstein's theory. The general,

non- linear theory is in any case needed for application to the more exotic context of neutron star deformations, as was discussed e.g. by Carter and Quintana [4]. Starting from this mathematically sound basis, we shall then proceed to the derivation of the weak field limit in two successive stages of approximation: we shall first impose the restriction that the gravitational radiation be weak, even though the unperturbed background field may still be strong (as in the case of a neutron star); finally we shall impose the condition that the background field also be weak (as in the case of the earth) so as to obtain the Dyson Papapetrou equations.

Unlike the treatments used by other authors such as those of the schools of Eringen (see e.g. Maughin [5]) or of Souriau (who was the first to set up the fully non-linear elasticity theory used here [6]) our present treatment will be based on the use of convected differentials, a powerful technique for the analysis of material media (elastic or otherwise) that generalises the more restricted convected differentiation procedure first introduced by Oldroyd [7]. In providing a self-contained introduction to the general concept of convected differentials and differentiation, and to their application to the theory of an elastic solid and its interaction with gravitation, the present article condenses several previous publications describing work carried out in collaboration with H. Quintana [8], [9], [10], [11], [12]. This article thus serves to update an earlier introductory survey by Ehlers [13] .

2 KINEMATICS OF A MATERIAL MEDIUM: MATERIAL REPRESENTATION

In Newtonian theory a material medium is usually visualised as a three dimensional manifold whose configuration at a given instant is specified by a non-singular mapping into three dimensional Euclidean space, the motion of the medium being given by the time variation of the mapping. However in General Relativity theory, where there is in general no canonically preferred set of three-dimensional sections (Euclidean or otherwise) of the fundamental four-dimensional spacetime manifold \mathcal{M} say, it is more convenient and natural to proceed the other way about. The medium itself is still to be conceived in the abstract as a three-dimensional manifold, \mathcal{X} say, (whose points represent idealised particles of the material) but its motion can be specified by a necessarily degenerate (four to three dimensional) mapping, \mathcal{P} say, of the world-tube traversed by the matter in \mathcal{M} onto the manifold \mathcal{X} , the world line of each idealised particle of the medium being projected under \mathcal{P} onto the corresponding point in \mathcal{X} .

We shall use Greek indices μ, ν, \dots to specify ordinary spacetime tensors on \mathcal{M} , and capital Roman indices A, B, ... to specify tensors on the manifold \mathcal{X} representing the medium. The spacetime indices μ, ν, \dots may be thought of in the traditional manner as specifying tensor components defined in terms of a local system of coordinates x^μ ($\mu = 0, 1, 2, 3$) on \mathcal{M} while similarly the material indices A, B, ... may be thought of as defined in terms of local coordinates X^A (A=1,2,3) on \mathcal{X} . However, following Penrose [14], it will also be convenient to interpret the index symbols in the abstract manner as merely an indication of the tensorial (or more general) character of the quantities concerned (rank, co/contravariant quality, and in the present case association with \mathcal{M} or \mathcal{X}) rather than as integers specifying concrete components. Thus with the indices

interpreted in this loose sense, the statement that a point with coordinates X^A in \mathcal{X} is the image under \mathcal{P} of a point with coordinates x^μ in \mathcal{M} may legitimately be expressed in a simple and natural manner by

$$X^A = \mathcal{P}\{x^\mu\} \quad (2.1)$$

whereas with a strict traditional interpretation of A and μ such an equation would be nonsense.

The operation \mathcal{P} of projection from \mathcal{M} onto \mathcal{X} evidently induces a corresponding projection, which we shall also denote by \mathcal{P} , from tangent vectors at any given point x^μ in \mathcal{M} onto tangent vectors at the corresponding point $\mathcal{P}\{x^\mu\}$ in \mathcal{X} . In an obviously natural notation we shall denote the projected image of a tangent vector by the same basic symbol, distinguishing the image from the original vector only by the appropriate change from Roman to Greek of the index symbol, i.e. for any spacetime tangent vector ξ^μ we shall set

$$\xi^A = \mathcal{P}\{\xi^\mu\}. \quad (2.2)$$

Due to the degeneracy of \mathcal{P} there is in general no corresponding induced projection operation for covectors except in the particular case of a covector α_μ that is orthogonal to the congruence of worldlines in the sense that

$$u^\mu \alpha_\mu = 0, \quad (2.3)$$

where the vector u^μ is tangent to the worldline at the spacetime point in question. In this particular case the projection will induce a corresponding covector on \mathcal{X} , for which as before we shall use the same basic symbol, i.e. we shall write

$$\alpha_A = \mathcal{P}\{\alpha_\mu\}, \quad (2.4)$$

and in this case the operation is reversible, i.e. we shall have a bijection

$$\alpha_\mu \leftrightarrow \alpha_A \quad (2.5)$$

between orthogonal covectors at x^μ in \mathcal{M} and covectors at $\mathcal{P}\{x^\mu\}$ in \mathcal{X} . We can use the (pseudo-) metric tensor $g_{\mu\nu}$ on the spacetime manifold \mathcal{M} (with signature $-+++$ and units such that $c = 1$) to define a corresponding bijection for tangent vectors by restricting our attention to tangent vectors on \mathcal{M} that are orthogonal to the world lines in the metric sense, i.e.

$$u_\mu \xi^\mu = 0, \quad u_\mu = g_{\mu\nu} u^\nu. \quad (2.6)$$

Subject to this restriction the projection (2.1) will have the same reversibility property as (2.4), i.e.

$$\xi^\mu \leftrightarrow \xi^A \quad (2.7)$$

in the same sense as (2.5).

The natural 1 - 1 correspondence that has just been defined between vectors or covectors orthogonal to the worldlines in spacetime and vectors or covectors in the material medium can evidently be extended directly to general orthogonal tensors as defined in terms of tensor products of orthogonal vectors and covectors. This correspondence is of vital importance for setting up any physical theory of the mechanical behaviour of the medium, since its spacetime evolution must be described in terms of geometrical quantities (tensors etc.) defined in spacetime, whereas its intrinsic properties can only be described in terms of quantities described directly in terms of the manifold \mathcal{X} representing the material medium (a requirement commonly dignified as the “principle of material objectivity”).

Since there is no guarantee that all physically relevant spacetime tensors will turn out to be automatically orthogonal to the worldlines, it is important to remark that they can always be canonically decomposed into a set of orthogonal tensors of equal or lower order by using the orthogonal projection tensor γ_ν^μ . In terms of the normalised unit tangent vector u^μ and the corresponding covector u_μ defined by

$$u^\mu u_\mu = -1 \quad (2.8)$$

this orthogonal projection operator is defined in terms of the unit tensor g_ν^μ by

$$\gamma_\nu^\mu = g_\nu^\mu + u^\mu u_\nu. \quad (2.9)$$

In the particular case of an ordinary tangent vector v^μ the natural decomposition into a set consisting of an orthogonal part ${}_\perp v^\mu$ and a scalar v^\parallel is given by

$$v^\mu = {}_\perp v^\mu + v^\parallel u^\mu, \quad (2.10)$$

where

$${}_\perp v^\mu = \gamma_\nu^\mu v^\nu, \quad v^\parallel = -u_\mu v^\mu, \quad (2.11)$$

while for a general covector ω_μ we similarly have

$$\omega_\mu = {}_\perp \omega_\mu + \omega_\parallel u_\mu, \quad (2.12)$$

where

$${}_\perp \omega_\mu = \gamma_\mu^\nu \omega_\nu, \quad \omega_\parallel = -u^\mu \omega_\mu. \quad (2.13)$$

This allows us to represent them in terms of sets of corresponding geometrical quantities defined at the corresponding point in the material medium in the form

$$v^\mu \leftrightarrow \{{}_\perp v^A, v^\parallel\} \quad (2.14),$$

and

$$\omega_\mu \leftrightarrow \{{}_\perp \omega_A, \omega_\mu\}, \quad (2.15)$$

the extension to general tensors of higher order being straightforward albeit cumbersome.

3 KINEMATICS OF A MATERIAL MEDIUM: CONVECTED DIFFERENTIALS

The concept of a material representation of spacetime tensors in terms of sets of tensors defined on the medium space (as defined by (2.14) and (2.15)) gives rise naturally to the concept of material variation defined as the difference between the material representations in any Lagrangian (i.e. worldline preserving) variation between different configurations, which may either arise from a mapping between different physically conceivable evolutions of the medium or else merely from a time displacement in a single evolution. The spacetime tensor corresponding to the infinitesimal material variation between nearby states of evolution of the medium will be referred to as the convected differential [12]. We shall use the symbol Δ to denote an ordinary Lagrangian differential and we shall use the notation $d[]$ for a corresponding convected differential. Thus in the case of a vector and a covector respectively, (2.14) and (2.15) give rise to the correspondences

$$d[v^\mu] \leftrightarrow \{\Delta_\perp v^A, \Delta v^\parallel\}, \quad (3.1)$$

and

$$d[\omega] \leftrightarrow \{\Delta_\perp \omega_A, \Delta \omega_\parallel\}. \quad (3.2)$$

To evaluate these expressions we use the fact that the world line preserving variation can only change the magnitude but not the direction of the unit tangent vector u^μ : explicitly

$$\Delta u^\mu = \frac{1}{2} u^\mu u^\nu u^\rho \Delta_{\rho\sigma} \quad (3.3)$$

where we use the abbreviation

$$\Delta_{\rho\sigma} = \Delta g_{\rho\sigma} \quad (3.4)$$

for the Lagrangian variation of the metric. In the case of a vector we obtain

$$\Delta v^\parallel = -u_\nu \Delta v^\nu - v^\nu \Delta u_\nu \quad (3.5)$$

$$\Delta_\perp v^\mu = \gamma_\nu^\mu \Delta v^\nu + v^\nu (u^\mu \Delta u_\nu + u_\nu \Delta u^\mu). \quad (3.6)$$

Since the bracketed quantity in the last expression is automatically parallel to u^μ it does not contribute to the material projection, so we obtain

$$\Delta_\perp v^A = \mathcal{P}\{\Delta_\perp v^\mu\} = \mathcal{P}\{\gamma_\nu^\mu v^\nu\} \quad (3.7)$$

and hence

$$\begin{aligned} d[v^\mu] &= \gamma_\nu^\mu \Delta v^\nu + u^\mu \Delta v^\parallel \\ &= \Delta v^\mu - u^\mu v^\nu \Delta u_\nu. \end{aligned} \quad (3.8)$$

In the case of a covector we obtain

$$\Delta \omega_\parallel = -u^\nu \Delta \omega_\nu - \omega_\nu \Delta u^\nu \quad (3.9)$$

$$\Delta_{\perp}\omega_{\mu} = \gamma_{\mu}^{\nu}\Delta\omega_{\nu} + \omega_{\nu}(u^{\nu}\Delta u_{\mu} + u_{\mu}\Delta u^{\nu}), \quad (3.10)$$

both terms in the last expression being automatically orthogonal to u^{μ} . Thus using

$$\Delta_{\perp}\omega_{\text{A}} = \mathcal{P}\{\Delta_{\perp}\omega_{\mu}\} \quad (3.11)$$

we immediately obtain

$$\begin{aligned} d[\omega_{\mu}] &= \Delta_{\perp}\omega_{\mu} + u_{\mu}\Delta\omega_{\parallel} \\ &= \Delta\omega_{\mu} + \omega_{\nu}u^{\nu}\Delta u_{\mu}. \end{aligned} \quad (3.12)$$

The extension to a general tensor is now automatic: it suffices to add an appropriately analogous term for each extra index. Thus for a general mixed tensor $T_{\nu\cdots}^{\mu\cdots}$ one obtains the relation [12] between convected and Lagrangian differentials in the form

$$\begin{aligned} d[T_{\nu\cdots}^{\mu\cdots}] &= \Delta T_{\nu\cdots}^{\mu\cdots} + T_{\rho\cdots}^{\mu\cdots}u^{\rho}\Delta u_{\nu} + \dots \\ &\quad - T_{\nu\cdots}^{\rho\cdots}u^{\mu}\Delta u_{\rho} - \dots \end{aligned} \quad (3.13)$$

where the Lagrangian differential of the covector u_{μ} is obtainable by substituting the formula (3.3) in the expression

$$\Delta u_{\mu} = g_{\mu\nu}\Delta u^{\nu} + u^{\nu}\Delta g_{\mu\nu}. \quad (3.14)$$

An important particular case [8] covered by the general formula (3.13) is that in which instead of making a comparison between nearby but different states of material motion (as one needs to do in perturbation theory) one wishes to study time variations in a single given state of motion. This corresponds to the case in which the Lagrangian variation is simply given by Lie differentiation with respect to a time displacement vector field, ζ^{μ} say, which we shall denote by $\zeta\mathcal{L}$, i.e. we set

$$\Delta = \zeta\mathcal{L}, \quad (3.15)$$

where the vector ζ^{μ} is an arbitrarily normalised tangent to the flow, which can therefore be expressed in the form

$$\zeta^{\mu} = u^{\mu}\Delta\tau, \quad (3.16)$$

where $\Delta\tau$ is an arbitrary scalar field interpretable as representing the local value of the corresponding infinitesimal proper time displacement along the world lines. Although the various terms in (3.13) will involve derivatives of ζ^{μ} , the intrinsic nature of the material variation will ensure that the convected differential will only depend on the scalar value of the displacement $d\tau$ at the point under consideration, so that it will take the form

$$d[T_{\nu\cdots}^{\mu\cdots}] = [T_{\nu\cdots}^{\mu\cdots}]d\tau \quad (3.17)$$

where the tensor $[T_{\nu\cdots}^{\mu\cdots}]$ so defined is what we refer to as the convected derivative. By direct substitution of (3.15) and (3.16) in (3.13) one can check that the terms involving gradients of $d\tau$ do indeed cancel out, so that one recovers the original formula [8] for the convected derivative, namely

$$\begin{aligned} [T_{\nu\cdots}^{\mu\cdots}] &= T_{\nu\cdots}^{\mu\cdots} + T_{\rho\cdots}^{\mu\cdots}(\dot{u}_\nu + \nabla_\nu)u^\rho + \dots \\ &\quad - T_{\nu\cdots}^{\rho\cdots}(\dot{u}^\rho + \nabla_\rho)u^\mu - \dots \end{aligned} \quad (3.18)$$

where ∇_μ is the usual metric covariant differentiation operator and where we use a simple dot without square brackets to denote covariant differentiation with respect to the proper time, i.e. $\dot{} = u^\rho \nabla_\rho$. In the particular case of the unit tangent vector itself, the dot operation gives the acceleration vector, whose covariant form is also expressible as the Lie derivative:

$$\dot{u}_\mu = u\mathcal{L}u_\mu. \quad (3.19)$$

Just as (3.13) is a generalisation of the earlier formula (3.18), so also (3.18) is itself a generalisation of a previous formula given by Oldroyd [7] for the particular case of tensors entirely orthogonal to the world lines, for which, as pointed out by Ehlers [13], the convected derivative reduces to the orthogonal projection of the Lie derivative. An important example is the divergence tensor $\theta_{\mu\nu}$ of the material flow, as defined by the decomposition

$$\nabla_\mu u_\nu = \theta_{\mu\nu} + \omega_{\mu\nu} - \dot{u}_\mu u_\nu, \quad (3.20)$$

where the vorticity tensor $\omega_{\mu\nu}$ is antisymmetric and $\theta_{\mu\nu}$ is symmetric. It is related to the strain tensor $\gamma_{\mu\nu}$ (i.e. the covariant version of the orthogonal projection tensor (2.19)) by

$$\theta_{\mu\nu} = \frac{1}{2}[\gamma_{\mu\nu}]^\cdot. \quad (3.21)$$

4 MECHANICS OF A PERFECT ELASTIC MEDIUM

The convected differential and derivative that were described in the previous section are potentially useful for the kinetic analysis of any kind of material medium. One of the simplest applications is to the theory of a medium whose behaviour satisfies the following (by now standard) criterion of perfect elasticity. A perfect elastic medium can be characterised succinctly by the condition that its energy - momentum tensor is a material function of the metric tensor with respect to the flow field specified by its timelike eigenvector. This means that the energy momentum tensor takes the form

$$T^{\mu\nu} = \rho u^\mu u^\nu + P^{\mu\nu} \quad (4.1)$$

where the pressure tensor satisfies the orthogonality condition

$$P^{\mu\nu}u_\nu = 0, \quad (4.2)$$

so that the material representation of $T^{\mu\nu}$ is expressible as

$$T^{\mu\nu} \leftrightarrow \{P^{AB}, 0, \rho\}, \quad (4.3)$$

and this representation must be a function of the corresponding representation

$$g_{\mu\nu} \leftrightarrow \{\gamma_{AB}, 0, -1\}. \quad (4.4)$$

Thus at each point in the three dimensional manifold \mathcal{X} representing the medium there are well defined functions determining the pressure components P^{AB} and also the density ρ in terms of the components γ_{AB} . There is however a restriction that prevents these functions from all being chosen arbitrarily, namely the local energy - momentum conservation law

$$\nabla_\mu T^{\mu\nu} = 0, \quad (4.5)$$

which has four independent components, whereas the acceleration of the flow has only three independent degrees of freedom. In order to avoid having an overdetermined system of equations of motion, one must require that the component of (4.5) along the flow (i.e. the conservation of rest frame energy as distinct from momentum) should be satisfied as an identity. The remaining independent equations are given by

$$\gamma_\nu^\mu \nabla_\rho T^{\rho\nu} = 0, \quad (4.6)$$

which is equivalent to the equations of motion

$$\rho \dot{u}^\mu = -\gamma_\nu^\mu \nabla_\rho P^{\rho\nu}. \quad (4.7)$$

The equation that must be satisfied identically is

$$u_\nu \nabla_\mu T^{\mu\nu} = 0, \quad (4.8)$$

which is equivalent to

$$\dot{\rho} = -(\rho \gamma^{\mu\nu} + P^{\mu\nu}) \theta_{\mu\nu}. \quad (4.9)$$

Now it follows from (3.21) that this last can be expressed in terms of convected derivatives as

$$[\rho] = -\frac{1}{2}(\rho \gamma^{\mu\nu} + P^{\mu\nu})[\gamma_{\mu\nu}], \quad (4.10)$$

which means that the convected variations along the world lines must satisfy

$$d[\rho] = -\frac{1}{2}(\rho \gamma^{\mu\nu} + P^{\mu\nu})d[\gamma_{\mu\nu}], \quad (4.11)$$

and hence that the corresponding variation of the material projections in \mathcal{X} must satisfy

$$d\rho = -\frac{1}{2}(\rho \gamma^{AB} + P^{AB}) d\gamma_{AB}. \quad (4.12)$$

This will be satisfied automatically for an arbitrary equation of state $\rho = \rho\{\gamma_{AB}\}$ if and only if the corresponding equations for the six algebraically independent pressure components are specified by

$$P^{AB} = -2 \frac{\partial \rho}{\partial \gamma_{AB}} - \rho \gamma^{AB}. \quad (4.13)$$

By carrying out a second partial differentiation of the single equation of state function for ρ with respect to the strain γ_{AB} we deduce that the material variation of the pressure tensor will be given by

$$dP^{AB} = -\frac{1}{2} (E^{ABCD} + P^{AB} \gamma^{CD}) d\gamma_{CD}, \quad (4.14)$$

where the elasticity tensor, whose material projection is defined by

$$E^{ABCD} = -2 \frac{\partial p^{AB}}{\partial \gamma_{CD}} - p^{AB} \gamma^{CD} \quad (4.15)$$

will obey the symmetry conditions

$$E^{\mu\nu\rho\sigma} = E^{(\mu\nu)(\rho\sigma)} = E^{\rho\sigma\mu\nu} \quad (4.16)$$

as well as the orthogonality requirement

$$E^{\mu\nu\rho\sigma} u_\sigma = 0. \quad (4.17)$$

A familiar special case of a perfectly elastic medium is that of an ordinary perfect fluid, which can be defined in the present context by the condition that its density be a function only of the determinant $|\gamma|$ of the material projection of the strain tensor, i.e.

$$\rho = \rho\{|\gamma|\}. \quad (4.18)$$

It follows from (4.13) that the pressure tensor will then take the purely isotropic form

$$P^{\mu\nu} = P \gamma^{\mu\nu}, \quad (4.19)$$

with the pressure scalar given by

$$P = -2|\gamma| \frac{d\rho}{d|\gamma|}, \quad (4.20)$$

while the elasticity tensor will be given in terms of the bulk modulus

$$\beta = -2|\gamma| \frac{dP}{d|\gamma|}, \quad (4.21)$$

by the formula

$$E^{\mu\nu\rho\sigma} = (\beta - P)\gamma^{\mu\nu}\gamma^{\rho\sigma} + 2P\gamma^{\mu(\rho}\gamma^{\sigma)\nu}. \quad (4.22)$$

5 SMALL GRAVITATIONAL PERTURBATIONS OF AN ELASTIC MEDIUM

Having seen how to set up a system of exactly self - consistent but non - linear equations governing a perfect elastic medium in the framework of General Relativity, we are now ready to derive the linearised wave equation governing small perturbations relative to a known background, such as might be induced by weak incoming gravitational radiation.

It is usually convenient to think of the perturbations as being determined in terms of a vector field ξ^μ that specifies the infinitesimal displacement of the worldlines relative to their positions in the known background space. Of course such a displacement is entirely gauge dependent and can always be reduced to zero by the use of an appropriate Lagrangian (worldline dragging) mapping of the perturbed space onto the background, but it is often convenient to fix the gauge by a more purely geometric requirement such as the preservation of a harmonic coordinate system, which is the generalisation of the usual newtonian procedure of defining the displacements relative to the fixed Euclidean structure of space. We shall use the symbol δ to denote the Eulerian variation of any quantity as specified by any such geometric prescription for the mapping of the perturbed spacetime onto the unperturbed background. The difference between the Lagrangian variations denoted by Δ and the Eulerian variations denoted by δ is given by Lie differentiation with respect to the corresponding infinitesimal displacement field ξ^μ , i.e.

$$\Delta = \delta + \xi\mathcal{L}. \quad (5.1)$$

Thus in particular the Lagrangian variation of the metric tensor is given by

$$\Delta_{\mu\nu} = h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}, \quad (5.2)$$

where we use the usual notation

$$h_{\mu\nu} = \delta g_{\mu\nu} \quad (5.3)$$

for the Eulerian variation of the metric arising from the gravitational waves under consideration.

As far as quantities characterising the material medium are concerned, it is easier to work with Lagrangian than Eulerian variations, since the former are related directly (via (3.13)) to convected variations and hence to the material variations that are governed directly by the equations of state. In the case of orthogonal covariant tensors, including the special case of scalars, we see from (3.13) that the Lagrangian variation is given directly by the convected variation, and hence from (4.12) we find that the Lagrangian variation of the density is given in terms of that of the metric by

$$\Delta\rho = -\frac{1}{2}\rho y^{\rho\sigma}\Delta_{\rho\sigma}, \quad (5.4)$$

where for future convenience we introduce the abbreviation

$$y^{\rho\sigma} = \gamma^{\rho\sigma} + \rho^{-1} P^{\rho\sigma} . \quad (5.5)$$

For a contravariant tensor however, (3.13) introduces extra terms, so that for the dependence of the Eulerian variation of the pressure tensor in terms of the metric (4.14) gives rise to the formula

$$\Delta P^{\mu\nu} = -\frac{1}{2} (E^{\mu\nu\rho\sigma} + P^{\mu\nu} \gamma^{\rho\sigma} - 4P^{\rho(\mu} u^{\nu)} u^\sigma) \Delta_{\rho\sigma} . \quad (5.6)$$

Using these results together with the expression (3.3) for the lagrangian variation of the flow field u^μ itself, we are now in a position to work out the perturbed equations of motion by taking the variation of the exact equations of motion (4.7). It is evidently most convenient to start from the Lagrangian variation, i.e.

$$\Delta(\rho \dot{u}^\mu + \gamma_\nu^\mu \nabla_\rho P^{\nu\rho}) = 0 , \quad (5.7)$$

which works out explicitly as

$$\begin{aligned} & (A^{\mu(\nu}{}_{\rho}{}^{\sigma)} - \rho y_\rho^\mu u^\nu u^\sigma) \Delta \Gamma_{\nu\sigma}^\rho + \gamma_\rho^\mu \epsilon_{\nu\sigma} \Delta_\tau E^{\rho\tau\nu\sigma} \\ &= \left(P^{\mu\nu} \dot{u}^\sigma - \frac{1}{2} \dot{u}^\mu P^{\nu\sigma} - 2A^{\mu(\nu}{}_{\rho}{}^{\tau)} (\theta_\tau^\rho + \omega_\tau^\rho) u^\sigma + \rho y_\rho^\mu \dot{u}^\rho u^\nu u^\sigma \right) \Delta_{\nu\sigma} \end{aligned} \quad (5.8)$$

using the abbreviation

$$\epsilon_{\mu\nu} = \frac{1}{2} \delta \gamma_{\mu\nu} = \frac{1}{2} \gamma_\mu^\rho \gamma_\nu^\sigma \Delta_{\rho\sigma} \quad (5.9)$$

for the relative strain tensor, and

$$A^{\mu\nu}{}_{\rho}{}^{\sigma} = E^{\mu\nu}{}_{\rho}{}^{\sigma} - \gamma_\rho^\mu P^{\nu\sigma} \quad (5.10)$$

for the modified elasticity tensor first introduced in a classical context by Hadamard. When the Lagrangian metric perturbation $\Delta_{\mu\nu}$ is evaluated by use of (5.2), and the corresponding perturbation of the affine connection components $\Gamma_{\nu\sigma}^\mu$ is evaluated using the corresponding substitution

$$\begin{aligned} \Delta \Gamma_{\nu\sigma}^\mu &= \nabla_{(\sigma} \Delta_{\nu)}^\mu - \frac{1}{2} \nabla^\mu \Delta_{\nu\sigma} \\ &= \nabla_{(\nu} \nabla_{\sigma)} \xi^\mu + \nabla_{(\nu} h_{\sigma)}^\mu - \frac{1}{2} \nabla^\mu h_{\nu\sigma} - \xi^\rho R_{(\nu\sigma)\rho}^\mu \end{aligned} \quad (5.11)$$

where the Riemann tensor is defined by

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \xi_\rho = R_{\mu\nu\rho\sigma} \xi^\sigma , \quad (5.12)$$

then we see that the basic perturbation equation (5.8) takes the form of a hyperbolic wave equation for the displacement vector ξ^μ when $h_{\mu\nu}$ is given. The characteristic cones and the corresponding sound speeds can be worked out directly from (4.7) by considering discontinuities [10] without any need of the full set of perturbation equations given here.

In order to have a complete system of equations governing the interaction of weak gravitational radiation with an elastic medium we need an additional wave equation governing the gravitational perturbation $h_{\mu\nu}$. This is obtainable by taking the appropriate perturbations of the Einstein gravitational equations, which are expressible (in units with $G=1$) by

$$\hat{R}^{\mu\nu} = 8\pi T^{\mu\nu} \quad (5.13)$$

where the Ricci tensor is defined by

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho \quad (5.14)$$

and where we use the notation $\hat{}$ for the partial trace subtraction operation defined by

$$\hat{R}^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R_\rho{}^\rho. \quad (5.15)$$

In our work up to this stage we have been concentrating on material aspects, so that Lagrangian variations have given the simplest formulae. However the advantages of being able to use more general Eulerian variations become apparent now that we come to the properly gravitational aspect, since it is well known that the perturbed Einstein equations in the Eulerian form

$$\delta(\hat{R}^{\mu\nu} - 8\pi T^{\mu\nu}) = 0 \quad (5.16)$$

can be greatly simplified by the imposition of the harmonic gauge condition

$$\nabla_\mu \hat{h}^{\mu\nu} = 0. \quad (5.17)$$

Under these conditions most of the terms drop out and one is left with an equation of the form

$$\square \hat{h}^{\mu\nu} = -16\pi \delta T^{\mu\nu} \quad (5.18)$$

where the relevant wave operator is defined by

$$\begin{aligned} \square \hat{h}^{\mu\nu} = & \nabla_\rho \nabla^\rho \hat{h}^{\mu\nu} - \hat{R}^{\mu\nu} \hat{h}_\rho{}^\rho + 2C^\mu{}_\rho{}^\nu{}_\sigma \hat{h}^{\rho\sigma} \\ & - \frac{2}{3}R_\rho{}^\rho (\hat{h}^{\mu\nu} - \frac{1}{4}g^{\mu\nu} \hat{h}_\rho{}^\rho) \end{aligned} \quad (5.19)$$

using the standard notation

$$C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} - 2g_{[\rho}^{[\mu} (R^{\nu]}_{\sigma]} - \frac{1}{6}g_{\sigma]}^{\nu]} R^{\tau}{}_{\tau}) \quad (5.20)$$

(with square brackets denoting antisymmetrisation) for Weyl's trace free conformal tensor. The Eulerian variation of the energy-momentum tensor is obtainable using (5.1) in terms of a Lie derivative and a more easily evaluable Lagrangian variation. Thus starting from (4.1) and using (3.1), (5.4), and (5.6) we obtain finally

$$\begin{aligned} \delta T^{\mu\nu} = & -\frac{1}{2}(\mathcal{E}^{\mu\nu\rho\sigma} + T^{\mu\nu}g^{\rho\sigma})\Delta_{\rho\sigma} \\ & + 2T^{\rho(\mu}\nabla_{\rho}\xi^{\nu)} - \xi^{\nu}\nabla_{\rho}T^{\mu\rho} \end{aligned} \quad (5.21)$$

where, following Friedman and Schutz [14] we have constructed a generalised (non-orthogonal) elasticity tensor with the same symmetry properties as those of the ordinary elasticity tensor (4.16) according to the prescription

$$\mathcal{E}^{\mu\nu\rho\sigma} = E^{\mu\nu\rho\sigma} + 6u^{(\mu}u^{\nu}P^{\rho\sigma)} - 8u^{(\mu}P^{\nu)(\rho}u^{\sigma)} - \rho u^{\mu}u^{\nu}u^{\rho}u^{\sigma}. \quad (5.22)$$

The coupled system of equations (5.8) and (5.9) simplifies considerably when not only the perturbations but also the background gravitational field is weak, as is the case in terrestrial (as opposed to neutron star) applications. In such cases we may suppose that there is a small dimensionless parameter, ϵ , loosely interpretable as an upper bound not only on the order of magnitude of the gravitational wave perturbations $h_{\mu\nu}$, $\epsilon_{\mu\nu}$ etc., but also of the deviations of the metric from the flat Minkowski form. It then follows from the Einstein equation that the density ρ must also be of linear order in ϵ as the latter tends to zero, while (by the virial theorem) the pressure in a self gravitating system is of even higher order, tending to zero even when divided by ϵ , i.e. in standard notation

$$P^{\mu\nu} = o\{\epsilon\}. \quad (5.23)$$

(In our original version [11] a printer's error substituted 0 in place of o throughout.) Since the unperturbed energy-momentum tensor will be at most of linear order in ϵ its perturbation will be of higher order, i.e.

$$\delta T^{\mu\nu} = o\{\epsilon\} \quad (5.24)$$

so that the gravitational wave equation (5.18) will to lowest order be of simple Dalembertian form, i.e.

$$\nabla_{\rho}\nabla^{\rho}h^{\mu\nu} = o\{\epsilon\}. \quad (5.25)$$

It follows that in addition to the harmonic gauge condition we can to this order make the further simplifications

$$h_{\mu\nu}u^{\nu} = o\{\epsilon\}, \quad h^{\nu}_{\nu} = o\{\epsilon\} \quad (5.26)$$

and to the same order the wave equation for the displacement reduces to the form

$$u^{\nu}u^{\sigma}\nabla_{\nu}\nabla_{\sigma}\xi^{\mu} - \rho^{-1}\nabla_{\nu}(E^{\mu\nu\rho\sigma}(\nabla_{\rho}\xi_{\sigma} + \frac{1}{2}h_{\rho\sigma})) = o\{\epsilon\} \quad (5.27)$$

in agreement with calculations of Dyson [2] and Papapetrou [3].

As a simple practical application Dyson considered the case of an elastic medium that is isotropic, as it will be in the case of a typical metal when considered on scales large compared with the microscopic crystalline domains. In such a case (in consequence of (5.23)) the elasticity tensor will be of the form

$$E^{\mu\nu\rho\sigma} = \beta\gamma^{\mu\nu}\gamma^{\rho\sigma} + 2\mu(\gamma^{\mu(\rho}\gamma^{\sigma)\nu} - \frac{1}{3}\gamma^{\mu\nu}\gamma^{\rho\sigma}) \quad (5.28)$$

where β is the bulk modulus and μ is the rigidity modulus. Now under these conditions it follows from (5.26) that the only gravitational term in (5.28) reduces to the form

$$\nabla_\nu(E^{\mu\nu\rho\sigma}h_{\rho\sigma}) = 2h^{\mu\nu}\nabla_\nu\mu, \quad (5.29)$$

which shows, as pointed out by Dyson, that the gravitational waves couple with the elastic displacement only via non-uniformities of the rigidity. (In the case of a traditional Weber bar detector the relevant non-uniformity is provided by the discontinuity at the surface of the cylinder.)

REFERENCES

- [1] Weber, J., *Phys. Rev.* **117** (1960) 306.
- [2] Dyson, F.J., *Astroph. J.* **156** (1969) 529.
- [3] Papapetrou, A. *Ann. Inst. H. Poincaré* **A16** (1972) 63.
- [4] Carter B., and Quintana, H., *Astroph. J.* **202** (1975) 54.
- [5] Maugin, G.A., *J. Math. Phys.* **19** (1978) 1198.
- [6] Souriau, J.M., *Géométrie et Relativité* (Herman, Paris, 1965).
- [7] Oldroyd, J.G., *Proc. Roy. Soc.* **A270** (1970) 103.
- [8] Carter, B., and Quintana, H., *Proc. Roy. Soc.* **A331**, (1972) 57.
- [9] Carter, B., *Commun. Math. Phys.* **30** (1973) 261.
- [10] Carter, B., *Phys. Rev.* **D7** (1973) 1590.
- [11] Carter, B., and Quintana, H., *Phys. Rev.* **D16** (1977) 2928.
- [12] Carter, B., *Proc. Roy. Soc.* **A372** (1980) 169.
- [13] Ehlers, J., in Israel, W., (ed) *Relativity, Astrophysics and Cosmology* (Reidel, Dordrecht, 1970) 89.
- [14] Penrose, R., in DeWitt, C., and Wheeler, J.A. (eds) *Battelle Rencontre* (Gordon and Breach, New York, 1968)
- [15] Friedman, J.L., and Schutz, B.L., *Astrophys. J.* **200** (1975) 204.